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Evaluation of lattice sums using Poisson's summation formula IV

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Abstract. Using Poisson's summation formula of dimensionality one or two, it is shown how a class of two-dimensional lattice sums involving modified Bessel functions can be reduced to rapidly convergent one-dimensional sums. The resulting formulae provide a simplification, and in many cases generalisation, of the ones obtained previously by Fetter and by Hautot.

1. Introduction

Recently we developed a method, based on the application of Poisson's summation formula, for the analytic evaluation of a class of lattice sums appearing in the theory of cubic lattices (Chaba and Pathria 1975, 1976a, 1976b; to be referred to as I, II and III, respectively). This method is now applied to a class of two-dimensional lattice sums, involving modified Bessel functions, which appear in the theory of clean type-II superconductors, rotating superfluid helium, and plasma oscillations in an array of filamentary conductors. The sums considered are the following:

$$X_\nu(\boldsymbol{\epsilon}; a) = \sum_{m_{1,2}=-\infty}^{\infty}{}' \cos(2\pi\epsilon_1 m_1)\cos(2\pi\epsilon_2 m_2)(m_1^2+m_2^2)^{-\nu/2}K_\nu[a(m_1^2+m_2^2)^{1/2}] \qquad (\nu=0,1,2,\ldots) \quad (1)$$

and

$$Y_\nu(\boldsymbol{\epsilon}; a) = \sum_{m_{1,2}=-\infty}^{\infty} [(m_1+\epsilon_1)^2+(m_2+\epsilon_2)^2]^{-\nu/2}K_\nu\{a[(m_1+\epsilon_1)^2+(m_2+\epsilon_2)^2]^{1/2}\} \qquad (\nu=0,1,2,\ldots) \quad (2)$$

where Σ' in (1) excludes the term with $m_1=m_2=0$. In (2), the vector $\boldsymbol{\epsilon}$ is generally non-zero and the summation Σ includes the term with $m_1=m_2=0$; however, if $\boldsymbol{\epsilon}=0$, then this term will be excluded from (2) as well and the resulting sum designated by the

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symbol Y'_ν instead of Y_ν. For reasons of symmetry, it will be sufficient to consider $0 \leq \epsilon_{1,2} \leq \frac{1}{2}$. Further, in view of the fact that

$$X_{\nu+1}(\boldsymbol{\epsilon}; a) = \frac{1}{a^{\nu+1}} \int_a^\infty z^{\nu+1} X_\nu(\boldsymbol{\epsilon}; z)\, dz$$

and

$$Y_{\nu+1}(\boldsymbol{\epsilon}; a) = \frac{1}{a^{\nu+1}} \int_a^\infty z^{\nu+1} Y_\nu(\boldsymbol{\epsilon}; z)\, dz,$$

it will suffice to consider the most frequently occurring case $\nu = 0$. At the same time, it will be convenient to denote the sums X_0 and Y_0 by the simpler symbols X and Y, respectively.

Some of the results obtained here are directly comparable with the ones recently reported by Fetter (1975) and by Hautot (1975), while others are new and, in general, exhibit a remarkably fast convergence.

2. Evaluation of the sum $X(\boldsymbol{\epsilon}; a)$

As pointed out by Hautot (1975), the Poisson summation formula of dimensionality two is not directly applicable to this sum because of the fact that the term with $m_1 = m_2 = 0$ is excluded from it. However, this difficulty can be circumvented by employing a technique due to Fetter *et al* (1966), as has already been demonstrated in II in connection with a different sum. Following that procedure, we obtain

$$\begin{aligned} X(\boldsymbol{\epsilon}; a) &= \sum_{m_{1,2}=-\infty}^{\infty}{}' \cos(2\pi\epsilon_1 m_1)\cos(2\pi\epsilon_2 m_2) K_0[a(m_1^2+m_2^2)^{1/2}] \\ &= \ln\left(\frac{a}{2\pi}\right) + \tfrac{1}{2}\gamma + \frac{1}{2\pi}B(\boldsymbol{\epsilon}) - \frac{a^2}{2\pi} \sum_{m_{1,2}=-\infty}^{\infty} [(m_1+\epsilon_1)^2 + (m_2+\epsilon_2)^2]^{-1} \\ &\quad \times \{a^2 + 4\pi^2[(m_1+\epsilon_1)^2+(m_2+\epsilon_2)^2]\}^{-1}, \end{aligned} \tag{3}$$

where γ is Euler's constant and $B(\boldsymbol{\epsilon})$ is given by equation (30) of II, namely

$$B(\boldsymbol{\epsilon}) = \lim_{a\to 0}[T(\boldsymbol{\epsilon}; a) - \pi \ln(1/a)], \tag{4}$$

where

$$T(\boldsymbol{\epsilon}; a) = \sum_{m_{1,2}=-\infty}^{\infty} [(m_1+\epsilon_1)^2+(m_2+\epsilon_2)^2]^{-1} \exp\{-a[(m_1+\epsilon_1)^2+(m_2+\epsilon_2)^2]\}. \tag{5}$$

Equation (3), which holds for all $a > 0$, is similar to, but considerably simpler than, the one derived by Fetter (1975) in that a bunch of terms involving complicated lattice sums, which appear in Fetter's expression and which finally depend only on $\boldsymbol{\epsilon}$, have been replaced by a single function $B(\boldsymbol{\epsilon})$ whose main properties have already been broached in II; see also equation (17) below. Recalling that

$$B(\epsilon \to 0) = (1/\epsilon^2) + C_2, \tag{6}$$

where

$$C_2 = \pi(\gamma - \eta) \qquad \eta = \ln\frac{[\Gamma(\frac{1}{4})]^4}{4\pi^3}, \tag{7}$$

we obtain from (3)

$$X(0;a)=\frac{2\pi}{a^2}+\ln\left(\frac{a}{2\pi}\right)+(\gamma-\tfrac{1}{2}\eta)-\frac{a^2}{2\pi}\sum_{m_{1,2}=-\infty}^{\infty}{}'(m_1^2+m_2^2)^{-1}[a^2+4\pi^2(m_1^2+m_2^2)]^{-1}, \quad (8)$$

in complete agreement with equation I(14). The asymptotic behaviour of the sums $X(\epsilon; a)$ and $X(0; a)$, as $a\to 0$, is readily obtained from the formulae (3) and (8), respectively.

To obtain an alternative expression for $X(\epsilon; a)$, we first split off the terms with $m_2=0$ and write

$$X(\epsilon;a)=2\sum_{m_1=1}^{\infty}\cos(2\pi\epsilon_1 m_1)K_0(am_1)$$

$$+2\sum_{m_2=1}^{\infty}\cos(2\pi\epsilon_2 m_2)\left(\sum_{m_1=-\infty}^{\infty}\cos(2\pi\epsilon_1 m_1)K_0[a(m_1^2+m_2^2)^{1/2}]\right).$$

The first sum is tabulated (see Gradshteyn and Ryzhik 1965, formula (8.526), 1). To the last sum we apply Poisson's summation formula in one dimension, whereby

$$\sum_{m_1=-\infty}^{\infty}\cos(2\pi\epsilon_1 m_1)K_0[a(m_1^2+m_2^2)^{1/2}]$$

$$=\pi\sum_{m_1=-\infty}^{\infty}\frac{\exp\{-m_2[a^2+4\pi^2(m_1+\epsilon_1)^2]^{1/2}\}}{[a^2+4\pi^2(m_1+\epsilon_1)^2]^{1/2}} \qquad (m_2\neq 0). \quad (9)$$

The summation over m_2 is now straightforward and the final result is

$$X(\epsilon;a)=\ln\left(\frac{a}{4\pi}\right)+\gamma+\frac{\pi}{(a^2+4\pi^2\epsilon_1^2)^{1/2}}+\pi\sum_{m=1}^{\infty}\left(\frac{1}{[a^2+4\pi^2(m-\epsilon_1)^2]^{1/2}}-\frac{1}{2\pi m}\right)$$

$$+\pi\sum_{m=1}^{\infty}\left(\frac{1}{[a^2+4\pi^2(m+\epsilon_1)^2]^{1/2}}-\frac{1}{2\pi m}\right)+\pi\sum_{m=-\infty}^{\infty}\frac{1}{[a^2+4\pi^2(m+\epsilon_1)^2]^{1/2}}$$

$$\times\frac{\cos(2\pi\epsilon_2)-\exp\{-[a^2+4\pi^2(m+\epsilon_1)^2]^{1/2}\}}{\cosh[a^2+4\pi^2(m+\epsilon_1)^2]^{1/2}-\cos(2\pi\epsilon_2)}; \quad (10)$$

it will be noted that in this expression only one-dimensional sums appear.

Three special cases of (10) will now be examined in detail.

2.1. $\epsilon=0$

Equation (10) now takes the form

$$X(0;a)=\frac{\pi}{a}\coth(\tfrac{1}{2}a)+\ln\left(\frac{a}{4\pi}\right)+\gamma+2\pi\sum_{m=1}^{\infty}\left(\frac{\coth[\frac{1}{2}(a^2+4\pi^2m^2)^{1/2}]}{(a^2+4\pi^2m^2)^{1/2}}-\frac{1}{2\pi m}\right). \quad (11)$$

The sum appearing here converges for all values of a. In the extreme case $a\to 0$, one may utilise the facts that

$$\lim_{a\to 0}\frac{\pi}{a}\coth(\tfrac{1}{2}a)=\frac{2\pi}{a^2}+\frac{\pi}{6}$$

$$\lim_{\epsilon\to 0}\sum_{m=1}^{\infty}\frac{\cos(2\pi\epsilon m)\coth(\pi m)}{m}=-\ln(\pi\epsilon)-\tfrac{1}{6}\pi-\tfrac{1}{2}\eta \qquad \text{III(8)}$$

and

$$\lim_{\epsilon\to 0}\sum_{m=1}^{\infty}\frac{\cos(2\pi\epsilon m)}{m}=-\ln(2\pi\epsilon). \qquad \text{III(32)}$$

Equation (11) then reduces to

$$X(0;a)\simeq\frac{2\pi}{a^2}+\ln\left(\frac{a}{2\pi}\right)+(\gamma-\tfrac{1}{2}\eta), \qquad (12)$$

in agreement with the leading terms of (8). This asymptotic behaviour could also be obtained directly from (10) by taking the appropriate limits whereby one gets

$$X(0;a)\simeq\frac{2\pi}{a^2}+\ln\left(\frac{a}{4\pi}\right)+\gamma+\frac{\pi}{6}+2\sum_{m=1}^{\infty}m^{-1}[\exp(2\pi m)-1]^{-1}.$$

Since

$$\sum_{m=1}^{\infty}m^{-1}[\exp(2\pi m)-1]^{-1}=\tfrac{1}{2}\ln 2-\tfrac{1}{12}\pi-\tfrac{1}{4}\eta, \qquad \text{III(A.1)}$$

we obtain the same result as in (12). Further, equating (8) and (11), we obtain the interesting relationship

$$\sum_{m=1}^{\infty}\left(\frac{\pi\coth(z^2+\pi^2m^2)^{1/2}}{(z^2+\pi^2m^2)^{1/2}}-\frac{1}{m}\right)$$
$$=(\ln 2-\tfrac{1}{2}\eta)-\frac{\pi}{2z}\left(\coth z-\frac{1}{z}\right)$$
$$-\frac{z^2}{2\pi}\sum_{m_{1,2}=-\infty}^{\infty}{}'(m_1^2+m_2^2)^{-1}[z^2+\pi^2(m_1^2+m_2^2)]^{-1} \qquad (13)$$

which is again valid for all values of z. The double sum appearing on the right-hand side of (13) can be expressed as a power series in z^2 by using the Hardy sums

$$\sum_{m_{1,2}=-\infty}^{\infty}{}'(m_1^2+m_2^2)^{-s}=4\zeta(s)\beta(s) \qquad (s>1). \qquad (14)$$

Equation (13) thus provides an explicit expression for the z dependence of the sum appearing on the left. At the same time, it expresses the double sum appearing on the right in terms of the single sum appearing on the left, which can be useful for large values of z.

2.2. $\boldsymbol{\epsilon}=(0,\frac{1}{2})$ *or* $(\frac{1}{2},0)$

In view of the exchange symmetry between ϵ_1 and ϵ_2, all results for $\epsilon_1=0$, $\epsilon_2=\frac{1}{2}$ must be the same as for $\epsilon_1=\frac{1}{2}$, $\epsilon_2=0$. Equation (10) then leads to the identity

$$\sum_{m=1}^{\infty}(-1)^{m-1}\frac{\coth[\frac{1}{2}(a^2+\pi^2m^2)^{1/2}]}{(a^2+\pi^2m^2)^{1/2}}=\frac{1}{2a}\tanh(\tfrac{1}{2}a)-2\sum_{m=1}^{\infty}\frac{\operatorname{cosech}(a^2+4\pi^2m^2)^{1/2}}{(a^2+4\pi^2m^2)^{1/2}}. \qquad (15)$$

This relationship holds for all values of a and reduces the slowly convergent sum on the left to the rapidly convergent sum on the right. As an example, we may consider the case $a = 0$ for which the sum on the left is known to be (see III(6*b*))

$$\left(\frac{3}{4\pi}\ln 2+\frac{1}{12}\right)=0{\cdot}248\,810.$$

On the right-hand side, the closed part itself gives 0·25 and the remaining sum (see III(6*c*)) contributes a paltry

$$\left(\frac{3}{4\pi}\ln 2-\frac{1}{6}\right)=-0{\cdot}001\,190.$$

Needless to say, the convergence will be even more rapid for $a > 0$.

2.3. $\epsilon \neq 0$, $a \to 0$

We now obtain

$$X(\epsilon; a) \simeq \ln\left(\frac{a}{4\pi}\right)+\gamma+\frac{1}{2\epsilon_1}+\epsilon_1^2\sum_{m=1}^{\infty}\frac{1}{m(m^2-\epsilon_1^2)}$$
$$+\tfrac{1}{2}\sum_{m=-\infty}^{\infty}\frac{1}{|m+\epsilon_1|}\frac{\cos(2\pi\epsilon_2)-\exp(-2\pi|m+\epsilon_1|)}{\cosh(2\pi|m+\epsilon_1|)-\cos(2\pi\epsilon_2)}. \tag{16}$$

Comparing (16) with the corresponding limit of (3), we obtain a very useful expression for the function $B(\epsilon)$, namely

$$B(\epsilon)=\pi(\gamma-2\ln 2)+\frac{\pi}{\epsilon_1}+2\pi\epsilon_1^2\sum_{m=1}^{\infty}\frac{1}{m(m^2-\epsilon_1^2)}$$
$$+\pi\sum_{m=-\infty}^{\infty}\frac{1}{|m+\epsilon_1|}\frac{\cos(2\pi\epsilon_2)-\exp(-2\pi|m+\epsilon_1|)}{\cosh(2\pi|m+\epsilon_1|)-\cos(2\pi\epsilon_2)}. \tag{17}$$

Setting $\epsilon_2 = 0$ and letting $\epsilon_1 \to 0$, we recover precisely the limiting behaviour (6). For $\epsilon_1 = \epsilon_2 = \frac{1}{2}$, we obtain

$$B(\tfrac{1}{2},\tfrac{1}{2})=\pi(\gamma-2\ln 2+2)+\frac{\pi}{2}\sum_{m=1}^{\infty}\frac{1}{m(m^2-\frac{1}{4})}-4\pi\sum_{m=-\infty}^{\infty}\frac{1}{|2m+1|[\exp(\pi|2m+1|)+1]}.$$

The first sum is equal to $4(2\ln 2-1)$ and the second sum is equal to $\frac{1}{4}\eta$; see III(A.4). Accordingly

$$B(\tfrac{1}{2},\tfrac{1}{2})=2\pi\ln 2+C_2. \tag{18}$$

Similarly

$$B(0,\tfrac{1}{2})=\pi\gamma-2\pi\ln 2+\pi^2-4\pi\sum_{m=1}^{\infty}\frac{1}{m[\exp(2\pi m)+1]}$$

and

$$B(\tfrac{1}{2},0)=\pi\gamma+2\pi\ln 2+4\pi\sum_{m=-\infty}^{\infty}\frac{1}{|2m+1|[\exp(\pi|2m+1|)-1]}.$$

The sum in the former expression is equal to $\frac{1}{4}(-5\ln 2+\pi+\eta)$ and the one in the latter is equal to $\frac{1}{4}(\ln 2-\eta)$; see III(A.2,3). In either case, we get

$$B(0,\tfrac{1}{2})=B(\tfrac{1}{2},0)=3\pi\ln 2+C_2. \tag{19}$$

3. Evaluation of the sum $Y(\epsilon, a)$

By a direct application of the Poisson summation formula in two dimensions, we obtain the remarkable result

$$Y(\epsilon;a)=\sum_{m_{1,2}=-\infty}^{\infty} K_0\{a[(m_1+\epsilon_1)^2+(m_2+\epsilon_2)^2]^{1/2}\}$$

$$=2\pi\sum_{m_{1,2}=-\infty}^{\infty}\frac{\cos(2\pi\epsilon_1 m_1)\cos(2\pi\epsilon_2 m_2)}{a^2+4\pi^2(m_1^2+m_2^2)}. \tag{20}$$

For $a\to 0$, equation (20) reduces to the asymptotic form

$$Y(\epsilon;a)\simeq\frac{2\pi}{a^2}+\frac{1}{2\pi}A(\epsilon), \tag{21}$$

where

$$A(\epsilon)=\sum_{m_{1,2}=-\infty}^{\infty}{}'\frac{\cos(2\pi\epsilon_1 m_1)\cos(2\pi\epsilon_2 m_2)}{m_1^2+m_2^2}. \tag{22}$$

The function $A(\epsilon)$ has been studied in detail in II and III.

To examine the case $\epsilon=0$, we split off the terms $m_1=m_2=0$ from both sides of (20), write $[a^2+4\pi^2(m_1^2+m_2^2)]^{-1}$ as

$$\frac{1}{4\pi^2}\left(\frac{1}{(m_1^2+m_2^2)}-\frac{a^2}{(m_1^2+m_2^2)[a^2+4\pi^2(m_1^2+m_2^2)]}\right).$$

and insert the limiting forms

$$\lim_{\epsilon\to 0}A(\epsilon)=-2\pi\ln(\pi\epsilon)-\pi\eta \tag{II(27)}$$

and

$$\lim_{\epsilon\to 0}K_0(a\epsilon)=-\ln(\tfrac{1}{2}a\epsilon)-\gamma.$$

Equation (20) then reduces to the expression (8) for $X(0;a)$, as expected.

To obtain a more rapidly convergent form for $Y(\epsilon;a)$, we apply the one-dimensional Poisson formula to the summation over m_1 and then carry out a straightforward summation over m_2. We thus obtain

$$Y(\epsilon;a)=\pi\sum_{m=-\infty}^{\infty}\frac{\cos(2\pi\epsilon_1 m)\cosh[(\frac{1}{2}-\epsilon_2)(a^2+4\pi^2m^2)^{1/2}]}{(a^2+4\pi^2m^2)^{1/2}\sinh[\frac{1}{2}(a^2+4\pi^2m^2)^{1/2}]}. \tag{23}$$

The original two-dimensional sum is thereby reduced to a one-dimensional sum. Incidentally, equation (23) could also be obtained directly from (20) by carrying out a summation over m_2 with the help of the formula

$$\sum_{m=-\infty}^{\infty} \frac{\cos(mx)}{m^2+\alpha^2} = \frac{\pi \cosh[(\pi-|x|)\alpha]}{\alpha \sinh(\pi\alpha)} \qquad (\alpha \neq 0). \tag{24}$$

For $\epsilon_1 = \epsilon_2 = \frac{1}{2}$, equation (23) reduces to the sum $S_2(0, 0, \frac{1}{2}a; 0)$ of Hautot (1975), while for $\epsilon_1 = 0$, $\epsilon_2 = \frac{1}{2}$ it reduces to his sum $S_4(0, 0, \frac{1}{2}a; 0)$. Further, equating $Y(0, \frac{1}{2}; a)$ and $Y(\frac{1}{2}, 0; a)$, we obtain a new identity

$$\sum_{m=-\infty}^{\infty} \frac{(-1)^m \coth(z^2+\pi^2m^2)^{1/2}}{(z^2+\pi^2m^2)^{1/2}} = \sum_{m=-\infty}^{\infty} \frac{\operatorname{cosech}(z^2+\pi^2m^2)^{1/2}}{(z^2+\pi^2m^2)^{1/2}}, \tag{25}$$

which may also be written in the form

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1} \coth(z^2+\pi^2m^2)^{1/2}}{(z^2+\pi^2m^2)^{1/2}} = \frac{\tanh(\frac{1}{2}z)}{2z} - \sum_{m=1}^{\infty} \frac{\operatorname{cosech}(z^2+\pi^2m^2)^{1/2}}{(z^2+\pi^2m^2)^{1/2}}. \tag{26}$$

Again, this relationship holds for all values of z and reduces the slowly convergent sum on the left to the rapidly convergent sum on the right; cf equation (15).

In passing we observe that the identity

$$\sum_{m=1}^{\infty} K_0[z(2m-1)] = \frac{\pi}{4} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(z^2+\pi^2m^2)^{1/2}}, \tag{27}$$

which follows a direct application of the Poisson summation formula and contains a slowly convergent sum on the right-hand side, can be rendered more useful if it is coupled with the new identity (25). We thus obtain

$$\sum_{m=1}^{\infty} K_0[z(2m-1)] = \frac{\pi}{2} \sum_{m=-\infty}^{\infty} \frac{1}{(z^2+\pi^2m^2)^{1/2}[\exp(z^2+\pi^2m^2)^{1/2}+(-1)^m]} \tag{28}$$

which converges rapidly for all values of z. Further, setting $z = k\pi$ ($k = 1, 3, 5, \ldots$) and summing over k with the help of formulae III(15) and III(A.4), we obtain the following sum in a *closed* form:

$$\sum_{\substack{k=1\\ \text{odd}}}^{\infty} \sum_{\substack{l=1\\ \text{odd}}}^{\infty} K_0(kl\pi) = \frac{4-3\sqrt{2}}{8} \zeta(\tfrac{1}{2})\, \beta(\tfrac{1}{2}). \tag{29}$$

Similarly, setting $z = p\pi$ ($p = 2, 4, 6, \ldots$) and summing over p with the help of formulae III(17) and III(A.2), we obtain another sum in a *closed* form:

$$\sum_{\substack{p=2\\ \text{even}}}^{\infty} \sum_{\substack{l=1\\ \text{odd}}}^{\infty} K_0(pl\pi) = \tfrac{1}{4}\ln 2 + (2^{1/2}/8)\zeta(\tfrac{1}{2})\beta(\tfrac{1}{2}). \tag{30}$$

These results (29) and (30) are more basic than the one reported by Hautot (1974), namely

$$\sum_{m=1}^{\infty} \sum_{\substack{l=1\\ \text{odd}}}^{\infty} (-1)^{m-1} K_0(ml\pi) = -\tfrac{1}{2}(\sqrt{2}-1)\zeta(\tfrac{1}{2})\beta(\tfrac{1}{2}) - \tfrac{1}{4}\ln 2, \tag{31}$$

since his follows simply by subtracting (30) from (29). Moreover, if instead we add (29) and (30), we obtain another result of this type:

$$\sum_{m=1}^{\infty} \sum_{\substack{l=1 \\ \text{odd}}}^{\infty} K_0(ml\pi) = \tfrac{1}{4} \ln 2 + \tfrac{1}{4}(2-\sqrt{2})\zeta(\tfrac{1}{2})\beta(\tfrac{1}{2}). \tag{32}$$

It seems of interest to point out here that some of the (three-dimensional) lattice sums studied in II and III have also been considered earlier by Sholl (1967). Comparing his equation (3.22) with our III(12*a*), we obtain yet another two-dimensional sum, involving modified Bessel functions, in a *closed* form:

$$\sum_{m=1}^{\infty} \sum_{\substack{p=2 \\ \text{even}}}^{\infty} K_0(mp\pi) = \sum_{m,n=1}^{\infty} K_0(2\pi mn) = \tfrac{1}{4} \ln (4\pi) - \tfrac{1}{4}\gamma + \tfrac{1}{2}\zeta(\tfrac{1}{2})\beta(\tfrac{1}{2}). \tag{33}$$

Combining (33) with (30) and (32), we further obtain

$$\sum_{\substack{q=2 \\ \text{even}}}^{\infty} \sum_{\substack{p=2 \\ \text{even}}}^{\infty} K_0(qp\pi) = \sum_{m,n=1}^{\infty} K_0(4\pi mn) = \tfrac{1}{4} \ln(2\pi) - \tfrac{1}{4}\gamma + \tfrac{1}{8}(4-\sqrt{2})\zeta(\tfrac{1}{2})\beta(\tfrac{1}{2}) \tag{34}$$

and

$$\sum_{m,n=1}^{\infty} K_0(mn\pi) = \tfrac{1}{4} \ln(8\pi) - \tfrac{1}{4}\gamma + \tfrac{1}{4}(4-\sqrt{2})\zeta(\tfrac{1}{2})\beta(\tfrac{1}{2}). \tag{35}$$

At this stage it is interesting to observe that, by utilising the theta-function transformation (Zucker 1974)

$$\sum_{m_{1,2}=-\infty}^{\infty} q^{m_1^2+m_2^2} = \theta_3^2(q) = 1 + 4\sum_{n=1}^{\infty} q^n(1+q^{2n})^{-1} = 1 + 4\sum_{m,n=1}^{\infty} (-1)^{m-1} q^{(2m-1)n}, \tag{36}$$

we can establish the following interesting identity:

$$\sum_{m,n=1}^{\infty} (-1)^{m-1} K_0[a(2m-1)^{1/2} n^{1/2}] = \tfrac{1}{4} \sum_{m_{1,2}=-\infty}^{\infty}{}' K_0[a(m_1^2+m_2^2)^{1/2}] = \tfrac{1}{4} X(0; a). \tag{37}$$

It follows that some of the results obtained in § 2 apply equally well to the sum appearing on the left-hand side of (37).

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